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Early Math in a Preschool Context: Spontaneous Extension of the Digital into the Physical

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Abstract. The paper presents a systematic examination of data from two early math interventions, involving 188 children aged 3,5-6,5 and their teachers. The aim is to cast light on how a digital early math game can be spontaneously extended into the physical environment by children and by teachers. Questions were: i) how the math content was extended to the physical room, ii) which elements of the game and the interventions inspired and provided affordances for the extensions iii) implications for children's learning of early math. The analyses revealed a great variety of ways in which children and teachers – acting on their own or together – brought the game out to the physical room. Among the underlying factors for this were everyone's experiences with the game, familiar narratives, and simple game design in terms of rules and visual features. Finally, positive influences on early math learning from the extensions were identified.

Keywords: early math, educational software, preschool, physical versus digital

1 Introduction

Over the past decade, the range of educational software has expanded rapidly, not least in the area of early math. The pedagogical value varies immensely between apps[2] and many are of questionable value [1, 2, 3, 4, 5]. Still, it is clear that use of certain apps can facilitate growth of early math competencies. Evidence for this can be found via a number of controlled intervention studies on preschoolers [6, 7, 8]. Some of the studies indicate that positive effects can remain and have an impact on future success – most notably for children from disadvantaged socioeconomic backgrounds [9, 10, 11].

Also before the digital era, teaching children about ordering, size and quantity comparison, sets, and numbers took place in preschool – by means of physical objects. The activities exploited both intentionally designed objects as Cuisenaire rods, Dienes

[2] "App" here is equivalent to 'digital application', including web-based software.

blocks and various Montessori materials; and everyday physical items as beads, shoes, toys, towels, cutlery, stones, trees.

With the arrival of digital tools, a set of questions arose for the area of early math[3]. How do the physical activities and materials relate to the digital ones? Are they separate or overlapping? What respective functions do they serve? (How) will physical activities and materials be affected as digital activities and materials become increasingly common?

This paper reviews the current state of research on the pedagogical advantages and disadvantages of digital vs. non-digital early math tools. It then analyzes data from two preschool interventions centered around a digital early math game with a focus on how teachers and children seem to bring content out of the digital game and into activities in physical space.

2 Background: Digital vs. Physical in Early Math

Before proceeding to the pros and cons of digital and physical materials in relation to early math, some basic concepts must be introduced. A *physical manipulative* is any object that can be manipulated – grasped or moved – and that potentially supports learning. Physical manipulatives for early math are, as said, of two kinds: those that are designed for the purpose and everyday objects that are not. A *digital manipulative* is a computer-based visual depiction of an object that can likewise be manipulated (*cf.* the definition of 'virtual manipulative' by Moyer et al. [13]).

A *representation* is something that points to (stands in for, signifies) something other than itself. Beyond its immediate physical nature, a physical manipulative – along with the activities it is used to engage in – serves as a representation. Seven pebbles represent the concept of seven; the act of removing two pebbles represents the concept "seven minus two equals five". Two toy cars simultaneously represent the concept of two, of pair, and of "a motor vehicle that human beings can enter and use for transportation". Three stripes on a piece of paper represent the number three, just as does the symbol "3".

Unlike physical manipulatives, digital manipulatives are always human created and generally purpose built. As a consequence, they are inherently representational, in the same way as drawings and films. A display of pebbles on a screen or in virtual reality – or projected via augmented reality – is a representation of pebbles. A note of caution: for a majority of people, at least in the West, a physical or virtual toy car represents an actual car; but for an individual who has never encountered a car, the representation does not work (or does not work fully or properly, even though there may be some overlap with the common cultural understanding). Something similar applies to the seven pebbles. For a one year old who plays with them, the pebbles do not represent anything – yet. She has not grasped the representational relation in the way that older children and adults nearly universally have. An important part of education consists in

[3] Arguably, research questions regarding physical tools and activities – how they work and why, and to what extent – only got addressed in any comprehensive way with the emergence of digital tools (see e.g. [12]).

facilitating the building of culturally established associations between the representation and the represented. Such understanding is central to learning. This raises important questions about the power – or lack of power – of different manipulatives to facilitate the process. In turn this leads to the concept of *affordance* [14, 15] and the ability of manipulatives to *afford* certain activities but not others. If sets of teddy bears, dinner plates or plums – physical or digital – entice a child into experimenting with them – say, placing one plate in front of each teddy bear, placing a plum on each plate and having two plums left – and if the child can be encouraged to describe what she has done, then these manipulatives have afforded activities beneficial to the development of basic math concepts. What a child *recognizes* as an affordance – the *perceived affordance* – depends both on the manipulative's design and on the child's interests and previous experiences [16]. These parameters are thus all essential to consider if the goal is to use manipulatives to support children's learning.

Another central notion in this paper is that of *extension from the digital to the physical*[4]. We have examined how math-related activities centered around digital manipulatives can be extended into physical space on the initiative from children and teachers. *Synchronic extensions* occur simultaneously with the use of the digital resource, *diachronic extensions* occur after – from a few moments to a few hours or even days. *Adjacent extensions* involve an obvious and direct mapping from the digital to the physical; in *distal extensions* the connection is looser, more associative or inspirational. Both pairs of concepts should be understood not in terms of strict bifurcation but as points along a continuum.

Two final points: first, the concepts of physical and digital themselves are not so well defined nor mutually exclusive as they might at first appear. Second, although manipulatives can and do facilitate learning, they do not *cause* learning to happen. Dienes [18] writes (p. 55): "one cannot over-emphasize that it is not the material itself which creates the true mathematical learning-situation". Rather, it is the actions made with or on the manipulatives – and reflections on those actions – that create learning [19, 20, 21, 22].

2.1 Benefits of the Digital in Relation to Early Math Learning

Kaput [23] was among the first to identify the ways that digital representations – *computer representations* as he called them – can help overcome limitations of physical resources. Physical representations, he argues, don't of themselves provide a record with which to examine and reflect on a previous activity. In contrast, digital representations readily do; e.g., a digital representation can include both an earlier scene of two balls coming from one side and three from the other, representing 2+3, *and* the present scene with the five balls together in a basket, representing the total 5. In addition, digital resources facilitate feedback. Using Kaput [23] as a starting point,

[4] A related but much broader concept is that of *extended digital interface* [17]. Interaction between a child and a digital resource necessarily involves a host of physical interactions: keyboard or tablet + fingers, earphones + ears, screen + eyes. The child may be engaged in conversation with other children either physically present or online. She is sitting somewhere. She may engage with, or refer to, other objects in the room. All these things make up the extended digital interface.

Moyer and colleagues [13] list other potential advantages such as adaptability, availability, ease of setup and shut down, and ability to print. They also point at how digital materials may break the stigmas and overcome many of the barriers that physically less able children face.

Several researchers [12, 24, 25] have argued that digital math resources provide better affordances when it comes to helping children focus on the "right" thing. Physical resources tend to have a host of extraneous features that, from a math perspective, can serve to distract rather than educate. Digital resources by their nature simplify, as they can strip away what is irrelevant. Features can then be added later, when the child's *zone of proximal development* [26] permits. Although similar solutions are possible with physical resources, they tend to be much more demanding and cumbersome to implement.

Finally, digital resources can be *pedagogically designed*: various forms of scaffolding, guidance and feedback – building on research findings into variation in learning trajectories – can be built into the resource [27] instead of requiring that a human instructor be present to provide them.

2.2 Benefits of the Physical in Relation to Early Math Learning

Manches and colleagues [28] describe, how physical objects and representations provide distinctive affordances for learning. Children can grasp objects to organize them, moving them closer or further away so as to remember whether they should be included in a task. This can only be approximated for digital space. A later article by the same researchers [12] develops this thought, arguing that tactile information helps a child focus by reducing demands on visuospatial memory: e.g., children can place their fingers on cubes to remember what to move next. More generally, both papers discuss the motor system's role in supporting cognition (see also [29]) and the role gestures play in problem solving and learning [30, 31]).

2.3 Which are Better for Early Math Learning: Physical or Digital Manipulatives?

A good number of researchers, e.g. [32, 33, 34]) hold up digital manipulatives as the winners. Kaput [23]) finds no evidence for the superiority of physical manipulatives. Manches and collaborators [12] in their review of digital and physical manipulatives conclude that the list of benefits identified for digital manipulatives is more impressive than the one for physical manipulatives.

However, several researchers question the meaningfulness of any such competition. Not the manipulatives themselves but the actions with and reflections on them create learning [19, 20, 21]. Mix [35] considers it well established that both physical and digital manipulatives – appropriately designed – can support children's early math learning, offloading memory and thoughts and helping generate actions and metaphors. For Uttal and colleagues [36] good manipulatives, physical or digital, are those that aid students in building, strengthening, and connecting mathematical ideas.

A robust early number sense requires an ability to handle a rich set of relations between objects and representations, from iconic to symbolic. Good early math activities, regardless of the kind of manipulative they use, draw upon children's physical environment – that they generally are strongly motivated to try to understand – such as how to share a bag of candy equally, how to decide how many bikes are needed for a group of children, or how to find a chair for each teddy bear when play acting.

The everyday world offers endless possibilities for children to learn the meaning of basic math concepts by asking, showing, explaining, discussing, and *doing*. In theory, it might be possible to develop a basis for mathematics without making direct use of physical movements, space, and objects; but children are embodied sociobiological creatures living in a physical world with which they are constantly interacting[5]. Early math concepts arise in that context: concepts of higher and lower, above and below, fewer and more, smaller and larger, taller and shorter, adding and removing, and so on. Even if children can learn aspects of basic math earlier, more quickly, and more smoothly through well-designed digital interventions, this in no way reduces the importance of physical interactions.

3 Original Concerns and Research Questions

Following the above, the most reasonable approach to early math education is one where both digital and physical manipulatives are made use of, and an establishment of adequate connections between the digital and the physical is in focus. However, there is no guarantee that this approach will be favored in preschools. A concern near at hand is, that the increasing introduction of early math apps in preschool might come at the expense of traditional math-learning activities with physical materials and that, as a consequence, the connections between the digital and the physical will not get sufficiently well established.

Preschool teachers' attitudes towards math complicate matters. Researchers [37] have shown that an important motive for many who chose the preschool profession is the assumption that neither interest nor competence in math is of high importance. Many have negative experiences of math from school. They do not think that they know math (even if they do), or they do not trust their ability to teach it. Compared to most other areas of preschool instruction, math is likely to evoke feelings of anxiety. One can imagine preschool teachers viewing educational math software as an opportunity for outsourcing math instruction.

We identified two sub-concerns: (i) What if introduction of digital early math activities means that activities with physical materials in physical space get lost? (ii) What if the mathematical concepts that children engage with in the digital sphere never make their way off the tablet or computer and into the children's everyday lives?

As it happened – at the same time that we engaged in discussions about these sub-concerns, we were towards the end of conducting two studies in which pre-school children played an early math app three-to-four times a week during six weeks. In these

[5] As noted earlier, the concepts of physical and digital, or virtual, are not so neatly separable as they might appear. The development of learning environments that incorporate virtual and augmented reality blurs the lines even further.

contexts, we made some observations of events where children, as well as teachers, spontaneously seemed to bring math content out of the digital game and into activities in physical space. We found this interesting and decided to examine the full documentation – video documentation and observational notes – from the two studies more systematically regarding to what extent and how a digital early math game can be extended into the broader social and physical environment. We formulated the following research questions:

1. (How) was digital math content moved out to the physical room – in other words, how was it extended into the physical preschool environment – by children and teachers?
2. Which elements of the game and the interventions inspired, scaffolded and provided affordances for children and teachers to make such extensions?
3. What were the implications of such extensions for children's learning of early math?

4 Method

As said, the empirical material originates from two intervention studies. Below we briefly describe: the interventions, the game, Magical Garden (MG), and the methods used to collect the data used for this article.

4.1 The Two Interventions

In both interventions, each child had her own tablet and used headphones. The children sat in small groups. The teachers, who received thorough training before the interventions began, organized the sessions and helped the children as needed.

In the first intervention [38] 42 children aged 3.9-6.5 years in four municipal preschools used MG two to three times per week in twenty-minute sessions over six weeks. The study focused on issues of adaptivity and feedback. Specifically, researchers and teachers observed how the game worked for children who were either ahead or behind in early math. How did the feedback provided in the game work? How well was it understood? How could it be improved? Researchers spent a total of twelve hours observing the children play the game and six hours interviewing or otherwise conversing with the teachers.

In the second intervention, named DIL [39] 146 4- to 6-year-olds in ten municipal preschools used MG every day or every other day in 15-20-minute sessions over six weeks for a minimum of 20 and a maximum of 30 sessions. DIL was one of the two experimental conditions in the research project [39]. In addition to the MG sessions, group-based exercises were intended to enhance the children's self-regulation abilities[6]. Their content was partly inspired by characters and narratives from the game: e.g., the

[6] The exercises were inspired by the Brain Train component of the intervention Parents and Children Making Connections - Highlighting Attention, developed by the Brain Development Lab at the University of Oregon [40].

bird character was used to train breathing: taking deep “bird breaths”. Posters on the walls described the group exercises and illustrated the self-regulation exercises and strategies with help of characters and other visual elements from MG. In this way, the DIL design included starting points for extensions of the digital into the physical. Note though that no materials in the self-regulation part of the intervention included any math content. Researches spent a minimum of one out of every five hours with each group of children, observing children and teachers while the children used the game and during other activities. 60 hours of video data were collected.

4.2 Ethical Considerations

Both interventions followed the Swedish Research Council’s ethical guidelines for research on human persons and were granted clearance by the ethics committee of the Karolinska Institute and by the Lund Ethical Review board (see www.epn.se). Guardians, teachers and children were, at each stage, informed and given opportunity to consent or withhold consent. Special attention was paid to children’s body language, including gestures and posture, since a child might not explicitly say “no” but indicate reluctance to continue by turning her face down. Anonymity was ensured by coding all personal information. The names used in this paper are pseudonyms.

4.3 The Digital Play-and-Learn Game Magical Garden

Magical Garden [41, 42]. builds on work by Griffin and Case [43]. On the first, most basic levels, the game involves neither symbolic nor iconic representations of numbers, beyond spoken number words. All actions are performed on virtual objects in virtual space. The game introduces such concepts as higher/lower, longer/shorter, too few/too many, and more/less, along with relations between them. A gradual linking of number words to magnitude and a variety of visual number representations follows, starting with iconic representations (e.g., a hand with fingers held up), progressing via the semi-symbolic (e.g., slashes as found in Roman numerals) to fully symbolic representations (Arabic numbers). The overall goal is to ensure that number concepts are well grounded and integrated with each other through a variety of forms of representation.

MG makes use of the pedagogical principle of learning-by-teaching with the child taking on the instructor’s role, helping a digital tutee (or *teachable agent* [TA]) [44] solve progressively more difficult tasks. The child is introduced to three characters – a mouse, panda, and hedgehog – whose garden is barren and in desperate need of watering. The child chooses one as her friend, whom she will help collect water drops to bring the garden back to life (see Figure 1). While the explicit narrative is about caring for the garden, the underlying purpose, of course, is practicing basic numeracy.

Fig. 1. Screenshots from Magical Garden showing the initial garden (left); the three characters: Panders the panda, Mille the mouse, and Igis the hedgehog (center); and a thriving garden (right).

MG comprises 60 scenarios, ordered by difficulty, defined by number range (1-4, 1-6, or 1-9), representation (fingers, dots, dice, number symbols) and method (counting, proto-addition/subtraction, "true" addition/subtraction). All scenarios can be presented through several sub-games that feature distinct narratives: e.g., a near-sighted bumblebee needing help to find the right flower or a treasure hunter needing help to reach one of several caves in a cliff by attaching balloons to her basket.

Regardless of sub-game, any given scenario is always repeated in three successive pedagogical modes (Figure 2): After having been introduced to the task the child practices on her own (mode 1); then the child shows her friend how to do the task (mode 2), and, finally, the child supervises her friend who attempts the task (mode 3). Supervising involves accepting the friend's answer (when judged correct) by pressing the happy smiley, or otherwise pressing the unhappy smiley (and, in some cases, also giving the correct answer).

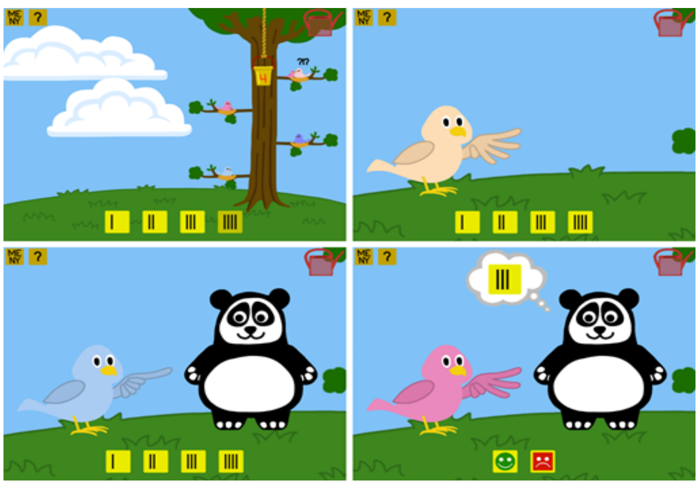

Fig. 2. Screenshots showing the pedagogical modes reflecting the TA paradigm, clockwise from upper left: overview of the sub-game Bird Rescue, where the task is to help the baby birds up into the tree to their parents; the child practices; the child shows her friend who watches; the child supervises her friend.

The game is adaptive in the sense that children are steered through the game by different paths. The amount of repetition, and the places where repetition takes place, varies. When a child masters a scenario, she moves on; when she encounters difficulty, she repeats the scenario – through varying sub-games – until her performance improves. If her performance remains low, the system goes back a half or entire level in difficulty so that the child can get more foundational practice. In this way some children get a substantial amount of training with a certain kind of task, whereas others quickly leave the same task behind.

The game is designed to be *inclusive* in the sense of adapting to different skills, *yet not* exposing children as being different from their peers – whether they are far behind or far ahead with respect to early math [45]. The game intentionally makes comparisons between children's competencies difficult: (i) The reward system (collecting water drops) is the same through all levels of play, with no additional rewards at higher levels. (ii) All successfully accomplished tasks yield the same number of waterdrops, independent of difficulty level; the number of collected drops reflects only the number of sub-games played successfully. (iii) The plants in the garden are randomly generated, so that all gardens look different regardless of how far one has proceeded. (iv) The different sub-games are determined randomly, which makes typical 'game level' comparisons difficult.

4.4 The Bird Rescue Sub-Game

The Bird Rescue sub-game (see Figure 2) will be used in some examples presented below. Therefore we describe it in some detail here: (1) The child is asked to help with a problem: a baby bird explains that it needs to get back to its parents after being blown off by a storm. The bird indicates with its feathers on which branch it has its nest. The tree has an elevator. If the correct button is pressed, the elevator brings the bird to its nest; otherwise it arrives at the wrong nest. (2) The child tries to solve the task by choosing among the labeled buttons at the bottom of the screen. Depending on difficulty level, the labels take the form of a hand (or two hands) holding up fingers, dots, dice, slashes (as in Roman numbers), numerals, or a mixture of these. (3) The child presses a button. Either the baby bird meets its parent, with laughter and hugs, or it ends up in another nest, with a funny noise and question mark; the bird provides feedback such as, "this is not my parent; I live a little higher up!" (4) The digital friend turns up and asks if she can watch the child play so that she, too, can help the baby birds. (5) After three rounds of observing, the friend asks if she can try. She suggests a solution, shown as a thought bubble containing a number representation. She asks: "am I thinking correctly now?" (6) The child accepts the solution or helps her friend understand her mistake and find the right answer. (7) When the digital friend and the child have helped the baby birds for a while, they are rewarded with water drops in their water jug. (8) They use the water to bring the garden back to life.

4.5 Data Collection

A variety of data collection methods were used in the two interventions: interviews, observations, field notes, focus-group discussions, informal conversations (with children and teachers), and video recordings – both while children were playing the game and doing other things. Note that none of the data collection was conducted to address the research questions in this paper as such. In the interviews and conversations with teachers, there was no direct probing into the occurrence of extensions by children or teachers. The guiding questions were instead: “how does the intervention with Magical Garden work out for your group of children?”, “what do they appreciate or dislike about the game?”, “how does it work for children who find basic math difficult?”, “how does the game work for children who are far ahead?”, “is there competition between the children in your group with respect to the game?”

5 Results and Analysis

The following summarizes our findings with respect to (i) ways in which children and teachers extended math content from the game into the wider preschool environment, (ii) elements of the game and the interventions that may have inspired, facilitated, or otherwise afforded the making of these extensions, and (iii) implications of the extensions for children’s learning of basic math. Results are grouped according to who initiated the extension: children, teachers, or both. The examples are based on a variety of data formats, but primarily video recordings, observations documented via field notes, and interviews and conversations with teachers documented via audio recordings or field notes.

5.1 Children Initiating Extensions

During both interventions, teachers gave abundant reports on extensions made by children (CE below stands for Child(ren)s Extension). According to these reports children often spontaneously engaged in what were (at least for the teachers) novel basic math activities outside the game but inspired by the game. Several teachers talked extensively about the children “playing” math and “talking” math more than before:

- “Playing Magical Garden has awakened a lot in the children; they play math more than before - like yesterday, some children played math with the toothpicks on the table. I think they imitated voices from the game, too.” -- CE-Ex1, preschool teacher.
- “Now the children talk math in a variety of daily activities; they initiate this themselves and go about like ‘my hanger is the first, Lisa’s is second, third, fourth, fifth!’” -- CE-Ex 2, preschool teacher. “Right, or like Jane, who pointed to a ladder we had in the yard and said: ‘my brother is seven’, then pointed downwards two steps and said something like ‘and I am five, that is smaller.’” -- CE-Ex2, preschool teacher.

- "At fruit time, we usually talk about how many children there are, how many apples or grapes we have and how to share them, like cut the apples in halves or go to the kitchen to ask if we can have so or so many more; now more children than before are eager to talk about the numbers and help solve the sharing. I mean they were interested also before in the sharing but not that many talked as much in number terms." -- CE-Ex3, preschool teacher.

The examples above illustrate distal and diachronic (not simultaneous with the game playing) extensions into everyday contexts. Some of the activities were obviously inspired by the game; others might be discussions that would have taken place anyhow, only now the teachers have taken notice, possibly due to an own awakened interest for math. The extensions were performed as embodied actions on toothpicks, fruit, hangers, and a ladder, as the children made associations back to the content of the game. The extensions created new affordances for the children to scaffold their learning [22].

Children not only used simple objects and representations of objects for the extensions. They found affordances in more complex content such as entire game activities. The following two observations involve clear connections between digital content and physical activities, thus exemplifying adjacent extensions.

- Children played that they were arranging a party, with several similar features as in the party sub-games in Magical Garden. In one of the sub-games there is a map that shows where birthday presents for the child's digital friend have been hidden by other characters in the game, and the child has to click ("walk") the exact number of steps marked on the map to get to the present. Children were observed to carry out similar activities in the physical room, with imaginary parcels; one child counted to a certain number, and another child should then walk as many steps to get to the parcel. -- CE-Ex4, researcher observation.
- At some of the preschools children were seen, shortly after having played MG, to be inspired by the robot character that now and then suddenly turns up in game scenes. The children walked and talked like robots and gave each other instructions on how to walk through an imaginative garden. -- CE-Ex5, researcher observation.

The voices and speech styles in the game seem to function as affordances for children's subsequent play – as well as for talking math.

The observations from teachers presented in the first tree examples above – CE-Ex1-Ex3 – include some evaluations of frequency change, proposing that the children bring math to everyday physical activities *more* than before, or "play" math *more* than before. Of course these assessments are subjective. Nevertheless, they are interesting given the earlier stated concern that the introduction of digital games would come at the expense of more traditional activities.

One final example, taken from a video recording of a group of four children playing MG during the fourth week of the second intervention, illustrates an adjacent and synchronic child-initiated extension. Emma (five years old) sits in front of a tablet lying on a low table. Like the others, she wears headphones and is logged into her personal account. A teacher is seated beside her and looks from time to time at what she is doing but does not assist in any way. As it happens, Emma has a green chalkboard behind her on the wall, not a part of the intervention, on which there are pre-printed letters (A-Z) and numbers (0123456789). While playing the sub-game Bird Rescue, she moves smoothly between the digital content and the (physical) chalkboard. We lack information on any previous experiences Emma might have had with the chalkboard or a number line of the kind printed on the board; but it is obvious from her actions that she is sufficiently familiar with the number symbols to use the board as an affordance to assist her counting – and her success in the game. She points with her finger at the numbers on the board, one after the other, touching them while counting out loud – something she could not do with the screen, where touching a number means *choosing* that button. Thus, Emma's use of the chalkboard for the combination of physical touch and counting, nicely complements the game.

Also, by her extension Emma ties together concepts from the physical and virtual worlds that point to the same underlying numerical concepts. She connects the set of feathers presented by the baby bird and the ordering of branches on the tree with her counting out loud and on her fingers, and she connects all of these with the symbolic representations on the chalkboard and the iconic and symbolic representations on the buttons on the screen. She turns her body back and forth between board and tablet, re-orienting it as needed while using her index finger to focus on the different representations.

Emma encounters a challenge in that the number line on the chalkboard includes the symbol "0" which is not included in the game. She had reached a quite advanced level in MG, so likely she was quite familiar with how the buttons in the game always begin with a representation of "one". Her pointing to the symbols "0", "1" and "2" on the board while saying "one, two... *it is three before the three"* is likely an attempt on her part to handle the apparent discrepancy. On the board there are three symbols – "0", "1" and "2" – "before the '3'".

5.2 Teachers Initiating Extensions

Not only the children created extensions into physical space. Several teachers commented on how they themselves had initiated novel math activities outside MG with inspiration from the game (TE below stands for Teacher's Extension):

- "When the children used the game, I became more aware of math in everyday activities; I did more sorting and counting with the children, and the children themselves more often counted spontaneously when they added or brought together things or material." -- TE-Ex1, preschool teacher.
- "When we were out on the yard, there was this small tree with its branches, and toys in different colors on the ground; I suggested to some children that the toys were creatures that wanted to go up in the tree, and we did three activities a little like in Magical Garden. At that moment I realized that there are many possibilities for early math activities around, and… I think I feel more certain that I can to this; it is really not difficult." -- TE-Ex2, preschool teacher.
- "We have been helped by this [intervention where children played MG] to make mathematics more visible both in other kinds of projects and I everyday practical situations … You know it is important because we have not been thinking in this way before… and… have realized that one does not have to choose! Digital

technology and practical everyday math activities do not exclude each one. It is not as if 'this is where the math is, in the math app'" -- TE-Ex3, preschool teacher.

All these examples represent diachronic and (more or less) distal extensions and show how MG provides affordances not only for the children's learning but also for the teachers' teaching. They reveal changes in thoughts and attitudes: an increasing awareness of math in everyday activities (TE-E1), an increased self-efficacy with respect to math skills (TE-Ex2), a different mindset on the relation between digital and physical with the epiphany that one does not have to choose between one and the other (TE-Ex3). Although these examples are too few to generalize, and there may well be (likely were) teachers whose attitudes to math were not affected positively, still, they offer tantalizing clues about what's possible.

One of the researchers observed yet another teacher-initiated extension; in this case, diachronic but (mostly) proximal. The second intervention closed with a party arranged, in one or other form, at each of the ten participating preschools. Each school had a wall calendar for the six weeks of intervention, and at the end of each calendar, the word "party" was written over a pennant-filled image from the game (Figure 3). All that was said in the introduction course for teachers was that it could be a good idea to end the six weeks with some kind of closure; nothing was said about what or how.

- At the majority of the preschools, the teachers prepared math-related aesthetical material for decorating a room for a party; paper cut into geometrical forms, balloons of different shapes, strings, and pennants, all reminiscent of MG's party sub-games. The children were engaged in decorating. Notably the teachers decided – without any instruction or prompt – to use math as a theme for the party. For example, they carefully divided the number of cookies equally between the children, poured half a glass of juice for each; they played games such as "dance-stop", where each child needs to sit down on a chair when the music stops, and there is one chair less than children, making use of the variety of different "counting situations" that arose during the games. The party was planned together by children and teachers, but sometimes the teachers also brought a surprise. At one preschool, three of the teachers dressed up as Panders, Mille and Igis. At another preschool, the party was held at night, accompanied by a festive dinner -- TE-Ex4, Researcher observations.

Fig. 3. On the wall in the preschools participating in the second intervention.

The next and final example of a teacher-initiated extension (TE-Ex5) represents a diachronic and adjacent extension. It comes from a video recording of the party where the teachers dressed up as the three digital friends (see Figure 4). Here the digital content of the game did not only afford extensions into pre-existing materials in the preschool. It also inspired to imaginary "make believe" and creativity among the teachers that far exceeded the researchers' expectations.

Fig. 4. Teachers dressed as Panders, Igis and Mille, in self-sewed costumes.

The teachers clearly took initiatives here with their self-made costumes – which required an investment both in time and effort – and with the mathematically oriented games that they set up. Which features of MG may have inspired and enabled these kinds of extensions? First, we suggest it was crucial that everyone involved was familiar with Mille, Panders and Igis; it would not have worked if only a few of the children had played the game or if all of them had played it but only on a few occasions. There was a common ground, and the characters were sufficiently interesting, meaningful and engaging to be moved into physical space in this particular way. Also crucial was the simplicity of the characters' visual design, which made the costumes practical. Indeed, the simplicity in shape, color etc. may well have inspired the idea to dress up, and the characters voices may have afforded and invited imitation of them.

Analysis: By extending the game's digital content into the physical space the teachers opened up novel affordances for early math. Games like musical chairs get the whole body involved; physical chairs cannot only be counted but also moved, grouped and regrouped, added and removed, while counting. By extending features from the digital context that the children were familiar with and recognized as math related, the children were given the chance to experience math with their bodies in states of emotional arousal. All together these activities afforded potentials for embodied cognition; scaffolding the co-activation of thinking, feeling and acting in relation to specific early math content [22] and by that grounding early math concepts in physical space and physical activities.

A birthday party theme such as the MG sub-games provide comes pre-equipped with positive emotions and ideas of play, having fun, and being together with friends. Extensive research shows [46, 47, 48] that emotional engagement facilitates learning.

5.3 Children and Teachers Initiating Extensions Together

In one preschool children and teachers had earlier created several wooden board games together. Now the children said they would like to create such games based on MG. This resulted in four adjacent (directly copying elements of MG) and diachronic extended versions that will be described below. It is impossible to know what inspired the children to suggest the idea or what led the teachers to agree to it, but likely the simplicity of MG and its sub-games in terms of appearance and rules is a key factor. Another is the way that MG had become part of their daily routine. Thus, children and teachers could probably imagine how the digital content could be represented in the physical world. The relation the children had built to the characters through game-play may have affected them to make their proposal.

The labyrinth game: In MG's labyrinth sub-game, children are to help a baby bird find her way out of the labyrinth and back to a birthday party. The child is presented with two numbers, represented iconically or symbolically, and asked to choose the larger. Only if the child chooses correctly does the bird proceed a bit further along the correct path.

- In the board game version, two children would take turns being "the player" and "the game", in several cases inventing new rules not fond in the MG version, while also discovering additional things to count: the number of bushes comprising the labyrinth walls or the number of steps to get out. They also created new labyrinths. -- C&TE-Ex1, Researcher observations.

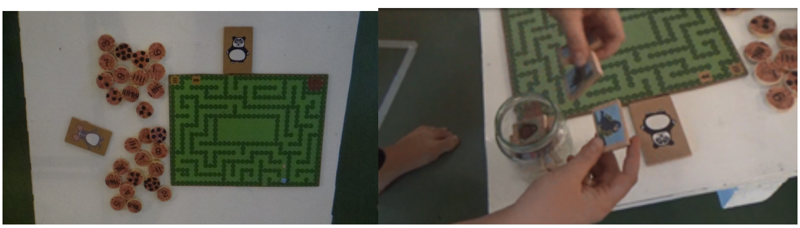

Fig. 5. The Magical Garden labyrinth game as board game.

The memory/domino game: This game combined classical memory and domino games. Mock-ups of the digital friends from MG, with different number representations painted on their bellies, were to be matched in pairs (flipping them over two at a time to reveal their numbers) or used as dominoes.

- Sometimes the children also used the bricks for other purposes, such as when one said to another, "Hi I am Panders, I am three years old; how old are you?"-- C&TE-Ex2, Researcher observations.

The Bird Rescue game: To a large extent, the Bird Rescue board game is a direct translation of the MG sub-game Bird Rescue. The teachers sawed wooden pieces in different geometrical shapes to which the children glued number representations in the form of dots, lines, or hands+fingers. The teachers manipulated the images of the birds' feathers to be counted more easily.

Two children would sit opposite each other. One took the ordinary "player" role from Bird Rescue. The other took on two roles: the digital friend and the baby bird needing help. Video recordings and direct observations show that the children attempted to imitate the MG characters, notably their voices. Sometimes they would fool with each other, just as they sometimes fooled with their digital friend while playing MG [38].

Fig. 6. The boardgame version of the MG sub-game Bird Rescue

The following example (C&TE-Ex3) presents an *incident* [49, 50] where the teachers introduce the board games they constructed with the children. The games are placed on low tables and the children are allowed to try them out. Teacher T is introducing the games to a group of the children, including a girl named Valerie.

Eight bricks are lying face up on the table; T turns the ninth face up: nine bricks, one for each number one through nine. The one T has just turned represents a six. The teacher (imitating the bird voice): *Can you help me to get home?* Valerie: *One, two, three, four, five*. Valerie counts with her finger along the ordered bricks with their number representations. Then she gives the wrong answer. The teacher says in a skeptical voice: *Is that five?* Valerie: *Yes*. T: *Mmmm?* Valerie: *One, two, three, four, five...* T: *That, you will place on top of it, there. Alright. And then I will try to go*. (T moves the bucket with the bird up the tree trunk.) *What did you say? Five?* Valerie: *Yes. Five*. T: *One, Two. Three. Four. Five. There. Here: look!* (T imitates the voice from the game.) *This is not where I live! I live higher up!* T moves the bucket down along the trunk. *Then it goes down again, and now you have to rethink. Think again.* Valerie (whispering): *One, two, three, four, five, six*. The numbers five and six are discussed. T: *Yes! OK! Which one is six, then? Oops. Well, which one is six?*

Valerie counts with her finger to the number six brick. She nods vigorously and takes the brick, puts it on top of the bucket. T: *Six. OK, then we will go to six*! Valerie is giggling and moving around on her chair. *One, Two, Three, Four Five Six*. T imitates the bird's voice: *There's where I live*!

Analysis: The board games represent adjacent extensions in that the narratives and characters, and many of the rules, come directly from MG; but there are also distal elements: new rules and new representations. The move from virtual to physical space necessitates certain changes: the board games permit no pre-programmed characters, and so all activities must be initiated by the children or teachers. Children took turns being the "digital" friend or the "player". They also engaged in a kind of pretend-play-act, where they prompted each other to answer back, such as for example telling their age when asked by the wooden teachable agent. This kind of pretend-play was often afforded by the teachable agent's voice, dialect and intonation in combination with the game's focus on math.

A set of manipulatives as in a board game, afford a range of physical and cognitive activities [14, 51]. Imitation – exploiting movement, voice, and facial expression – is clearly in play in the Bird Rescue example, affording new opportunities for training in basic math skills. The teacher uses the exact same words as the bird in the sub-game, which is possible because she has played the game herself quite a few times. With her hands, she guides the game pieces through the same movements as in the sub-game. (The child is using imitation as well – in a way, imitating the teacher who is imitating the bird. The child giggles and moves around when the baby bird rejoins its parent, just as the bird twitters and moves around.) What are not copied over are the background music and the meditative atmosphere of the sub-game; both children and teachers move more quickly counting, assessing, and interacting with each other.

The teacher in the example above (C&TE-Ex3) extends the game's multimodal representational nature with multimodal representations of her own. She uses oral questioning and repetition to assist Valerie when Valerie gives a wrong answer, at the same time she employs body language – something that the character in the digital game is ill-equipped to do. She writes and draws to make the number representations clearer. She and Valerie can both make use of tactile information – e.g., the feel of the wooden bricks – in a way that is not possible in the digital game. The teacher uses her hand to show how high or low a number is and to highlight the relation between numbers, perhaps at least unconsciously aware of the role that gestures play in problem solving and learning [30, 31] – especially for preschool children whose grasp of even the most basic mathematical notions remains limited. The example diverges from a standard pedagogical context in which the child gives an answer that the teacher assesses. The teacher is still assessing the answer, but she does so as a teacher pretending to be a baby bird. At this specific preschool, the teachers would often go in and out of roles in pretend play, and the children seemed to accept and appreciate this kind of "math play". Perhaps it is less intimidating, or at least fun for a change, to have a bird assess you rather than your teacher.

Something should be said about the advanced metacognitive thoughts and complex shifts of perspectives these children exhibit while interacting with the board games. Preschool children are often thought to be unable to take others' perspectives: a preconception that these examples challenge.

It is striking how children's interaction with the board games both is similar and dissimilar to their interactions with the corresponding sub-games in MG. They try to find ways to cooperate in both the digital and physical versions, even manipulating the game to do so, even though that is a lot harder to do in the digital case. They want to play with their digital friends, something that in some ways is much easier to do with the board games since the characters are printed on wooden tiles; but they talk more to their digital friends more in the digital versions, probably because the characters seem more alive there: they move around and makes sounds.

Another example (C&TE-Ex4) of teacher-and-child-initiated extension comes from a birthday party during the second intervention (Figure 9). The children who played MG demonstrated the game and talked about it to younger peers who had not played.

- The room was decorated with pennants and balloons, just like in the party in MG. The children told the peers which digital friend they had chosen They explained how they taught their friend what they had just learned but also how they sometimes fooled their friend They mimicked the various characters' voices and utterances, such as "What a good teacher you are!" which is something the teachable agent says when the child has guided her/him well (each time that the child correctly shows the digital friend three times in a row how to solve a task). The children told their peers how different numbers can be represented and how to walk a line counting the steps/numbers horizontally, they talked about the balloons in the game and showed with their arms and a real physical balloon how to count levels vertically. The presentation was initiated by teachers who wanted the invited children to get to know what had been going on in the intervention activities, but exactly what the children should present to their invited peers had not been practiced beforehand. The teachers were amazed, particularly with how the children used early math concepts such as higher/lower and more/less in the demonstrations to the peers. -- C&TE-Ex4, Researcher observations.

That the children were familiar with the learning-by-teaching model and had practiced a teacher role likely facilitated the activity. It may have worked as an affordance both for teachers to didactically stage for this peer-to-peer teaching-and-learning event, and for children to be more comfortable in the teacher role and more generally self-efficacious. This might be particularly important for those children used to taking more passive roles and *not* acting spontaneously as teachers at home (for younger siblings) or in other contexts.

A final example drawn from both researcher observations and conversations with teachers involves a puppet theatre, found by researchers at one of the preschools. The theatre was constructed from a large box and depicted a familiar MG scene with the three digital friends, as laminated figures on flower sticks. The teachers said that the theatre had been constructed on the children's initiative and that they used it to play theatre to each other, mimicking voices and characters from the game. The teachers also expressed surprise:

- "That the children bring this out from the digital… In the game they have this "friend" and that they then seem to find it natural to bring her out and make her physical… We found this cool and surprising; it was not at all something we had expected."-- C&TE-Ex5, preschool teachers.

6 Conclusion of Results and Analysis

Our observations and analyses reveal a variety of ways in which the children and teachers – acting on their own or together – extended MG into the wider preschool environment. Several teachers explicitly remarked that they found it easier to initiate early math learning activities in the physical environments after as opposed to before the digital intervention. As researchers, we were made aware of the power of influences between children and teachers. With only modest encouragement from their teachers, the children easily picked up on new concepts and expanded the digital into their everyday physical interactions at the same time that the teachers were keen on picking up the children's creative extensions. The number of ways that the computer-based math activities spilled over is nothing short of surprising. Clearly, a digital math intervention need not come at the expense of early math activities in physical space.

Our two original concerns appeared to be disconfirmed: namely that (i) the mathematical concepts that children engage with on the computer might never make their way off the computer and into the children's everyday lives, and that (ii) physical activities with respect to early math might get lost. At the least, we have clear evidence that none of these things *must* occur.

We have, then, attempted to identify factors in the design of MG and the setups of the two interventions that may have invited or facilitated the extensions we describe. Among these are the common ground in that all of the children and teachers were familiar with MG; that the game employs a large number of narratives; that it represents many basic, everyday objects and actions; that its design is simple, both visually and in terms of its rules.

It is likely the case that some children found it harder than others to bring the digital content into their everyday lives and certainly possible that some teachers *did* feel relieved not to need to be teaching basic math themselves, now that the computer could do it for them, or at least felt that they could to some extent tick this off. Given the qualitative and small-scale nature of our study, with a limited number of examples, it is impossible to make any quantifiable claims or generalizations.

7 Discussion

7.1 Methodological Limitations

The sample size in the study is limited. Even though about 150 children and 30 teachers participated, they came from only 14 classrooms. There were important differences between the two interventions, which might make their extensions not directly comparable. The first intervention had only one component: the game; whereas the second intervention added a second component focusing on self-regulation training. In some cases, it may be difficult to disentangle the influence of the two components on a particular instance of extension.

Talk of synchronic/diachronic and adjacent/distal extensions necessarily involves interpretation and speculation. Consider the children instructing each other how to walk in the find-the-parcel activity, or the way they imitated a robot walking. Since the video-recorded activities occurred just after a game session, we conclude that the MG sub-game on finding hidden parcels (by walking in a "correct" way) and the robot character from the game were the direct inspiration. Having said that, computational thinking and even simple programming ("coding unplugged") are appearing in some Swedish preschools. As far as we know, the preschools we studied had not introduced these subjects, but of course it is impossible to know whether children may have encountered them elsewhere. Some children have experience playing with robots; many more have experience pretend playing robots, commanding them to walk with directions like "four steps forward, six steps to the left…"

The very nature of the analysis – a *post hoc* analysis on data collected in studies designed with other research questions in mind – calls for caution. We are primarily making suggestions but have found some tantalizing clues and possibilities worth further, systematic investigation.

7.2 Leaving the Physical/Digital Dichotomy Behind

There is fairly strong evidence in the form of controlled and large-scale studies that some digital interventions can and do support the development of early math skills in children. For obvious reasons fewer studies have looked at traditional pedagogical methods the same way – here large-scale controlled studies are much more demanding – still, there is other kinds of evidence for their value. In particular, there is no evidence to recommend one replacing the other – or even taking priority over the other (as some [52, 53, 54, 36] would suggest that traditional, non-digital, methods should do).

The idea that the physical material is the obvious 'concrete template' that has to precede the more abstract and digital material, is often traced back to Piaget [55] Montessori [56], and Bruner [57] theorists who all focused on the concrete nature of children's thought. Such historical projection is tempting but misguided. No one can know how these scholars would have reasoned had they experienced the digital revolution as contemporary generations have.

Recent years have seen substantial changes in people's approaches to and understandings of the digital and the physical. For those who grew up before personal computers, paper-based versions of crossword puzzles, tic-tac-toe games, etc. came first and are the templates for their digital counterparts. Children growing up today, amidst augmented and mixed reality, see things differently. Consider the recent phenomenon of *Pokemón Go*, where, interestingly enough, the collectable cards familiar to an older generation have seen a revival.

Research suggests that contemporary children do not separate the digital from the physical the way that grownups do; they may be seen to regard them as different dimensions of a unit. People talk of a natural bond between children and technology, and the expression "digital natives" has entered the common lingo. Adults are seen as the digital immigrants [58]. Some researchers though think the picture is not so clear [59]. Certainly, children can appear fearless when confronting complicated digital interfaces and they happily embrace the Internet with all its software and all the possibilities it provides to explore, but it does not follow that they understand technology automatically.

Teachers often use digital activities to make children feel more comfortable and self-efficacious in their environment. Jigsaw puzzles demand fewer motor skills to assemble digitally. Real insects can be less frightening when first introduced through a digital interface. For preschool children's encounters with early math phenomena, it is not self-evident that the building of a concept always has to start out by establishing an understanding for the concept while using physical material in order to take a next step and understand the concept also while using digital materials.[7] The relation between digital and physical is ultimately synergistic. The embodied cognition literature (e.g., [60]) describes (adult) thought as embodied in previous perceptual experiences, providing yet one reason to question those who would claim that children's cognitive development is really best described in terms of increasing independence from the concrete and physical.

7.3 Individual Variability

Providing a variety of activities employing physical, digital, and mixed manipulatives gains greater importance in view of the huge individual variation in how children go about developing early math skills. These skills are not well-delimited chunks of understanding that arise neatly at one point in time. Instead one finds chains of embodied cognitive testing, practicing, repeating, expanding, and reflecting [22]. Also, the idea of *one* generic learning trajectory – or even a readily countable set of learning trajectories – has become untenable [27, 22].

An example from our data: some of the children struggled when the iconic representations – dots and slashes – were introduced in MG. Their teachers engaged them with *Kapla* building blocks, which worked out well because they could then go on and make progress in the game. It would not have made sense to have the entire class stop their MG play to engage with the blocks (nor would that have worked within the framework of the intervention).

There is variability not only in skills and trajectories but also in what motivates children. Some children are more readily engaged by hands-on activities; others take more easily to tablets and screens. Based on our observations, it was clear that the level of emotional engagement evoked by different sub-games in MG varied. At the group level, the sub-game with a tractor and crane loading fruit onto a truck did not spark much engagement and was quite frequently considered boring. Yet for a few children, it was their favorite sub-game.

This motivational side of affordance should not be forgotten. In the end, a child must find an activity or manipulative meaningful if she is to learn from it.

[7] Note that we are not discussing young infants. There is a high degree of consensus that an infant needs experiences with physical grasping and pointing at objects – in general, needs to develop her sensorimotor skills – before she can profit from (or should be introduced to) digital devices.

7.4 Everyday Emotional and Multimodal Affordances for Extension

Of course there are possibilities for the contents of a math textbook to extend into the wider environment as well. The question is why this does not happen more often. We suggest that the multimodality of digital learning games – such as MG, with animations, music, and speech – affords and encourages these extensions, in part by scaffolding the children's (and teachers') learning. MG's didactic design – where children learn by teaching – provides a crucial affordance, as does the way and that children build an emotional bond to their chosen digital friend and other characters in the game that they help in various ways. Since that help is, without exception, math related, it is not surprising that the extensions are, too. We also think that MG's everyday themes help and that gardening or playing with birds or bumblebees makes an easier springboard for extension than exploring space or encountering ghosts. Many teachers said that it is in the most everyday of activities – cutting fruit or taking a walk – that they come to think about math and talk math and play math with the children.

Manches and colleagues [12] discuss how the tactile information in physical objects is important for children's learning. Others, e.g. [12] have explored the possibility to enhance digital resources with tactile information – vibrations, warmth, and texture – either through actual tactile ("haptic") feedback or visually. Although it remains hard to embrace the tactile in digital learning, in various ways children and teachers obviously can find ways to add it on their own, through extensions they create.

Indeed, as this paper argues – and with plenty of support from related research (e.g., [19, 20, 21, 22]) – it is often largely in acting and reflecting on one's actions that learning takes place. In the context of our two interventions, we believe learning takes place in the creative makings inspired from the digital game. The same researchers cited above emphasize a point we made earlier: that the real potential for learning lies not in the original activities but in reflecting on them and engaging in further activities– which we have exemplified in this paper. The extensions thus created might not have the universally perfect design, but they can be what Mix [35]) calls "adequately designed": being made not for children in general but for the particular children at hand.

7.5 Children's Social Nature as Further Impetus for Extensions

Even though MG is designed for individual use, the preschoolers in the two interventions found ways to make it a social affair, particularly in their extensions to the game but also in the game itself. There are, indeed, a lot of relations going on both within the game and in the extended digital interface [61].

One of the characteristics of play that we observed was imitation of the characters from the game. The children imitated everything they could about the characters: their voices, speaking style, walking style, facial expressions, and actions (e.g. arranging a party for a friend). The children (and their teachers) dressed up, fooled each other, helped each other, gave each other compliments, and asked each other questions, just as the game characters did.

The metacognition and changes in perspective that we observed are scaffolded [22] through interactions with teachers, fellow classmates – and digital friends.

7.6 Future Studies

It is important to embrace both physical and digital to harness the potential synergy between them and realize the full potential of extensions for teaching and learning. We have shown that spontaneous extensions of a digital game into the physical environment, on the initiative and teachers alike, can and do occur. We have been inspired to develop new digital content taking advantage of what we have learned to (re-)focus pedagogical strategies for early math learning. In a follow-up study, we intend to develop extensions of the digital content beforehand and make them available side by side with the digital content. Ideas for costumes, face painting, and scripted role play could be provided, along with suggestions for how the teachers could take on the role of the teachable agent themselves. The goal is a design that begins from a well-crafted digital learning environment where the environment itself, and the supporting materials that are provided with it, are explicitly designed to facilitate spontaneous extensions. One possibility for strengthening the affordances in MG to promote such extensions would be to include prompts, directed towards the physical environment [16] to which the child should respond: e.g., the digital friend could ask the child to look around the classroom for representations of numbers or to collect objects while counting steps in the game. Pre-designed extensions can offer guidance, whereas spontaneous ones may convey more direct meaning and be more motivational. Our goal is to combine the advantages of both.

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